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Kodaira dimension of holomorphic singular foliations

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Abstract. We introduce numerical invariants of holomorphic singular foliations under bimeromorphic transformations of surfaces. The basic invariant is a foliated version of the Kodaira dimension of compact complex manifolds.

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Mathematical subject classification: Primary 32L30, 58F18; Secondary 14J26, 32H04.

1 Introduction

This paper is concerned with holomorphic foliations with a finite singular set on compact complex regular surfaces and their modifications under bimeromorphic transformations. Our aim is to define foliated analogues of the numerical bimeromorphic invariants of compact complex manifolds and begin a bimeromorphic classification of foliations.

A *bimeromorphic* transformation $\phi : M - \to N$ between compact complex manifolds is a meromorphic mapping whose restriction $\phi_{|\, M-\Sigma} : M-\Sigma \to N-S$ is a biholomorphism, for Σ and S analytic subsets (when M and N are projective manifolds, ϕ is called a *birational* transformation). Such transformations are quite abundant for compact complex manifolds and very natural in the theory of holomorphic singular foliations. In the local theory (see e.g. [C-S]), the *blow up* $\phi : M \to N$ of a singularity p of a foliation; $\Sigma = \phi^{-1}(p)$ is the exceptional line and $S = \{p\}$. In the global theory (see e.g. [LN], [Br1]), when there are

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different compactifications $M = \mathcal{U} \cup \Sigma$ and $N = \mathcal{U} \cup S$ of an algebraic foliation of an affine surface $\mathcal{U}$. We say that a foliation $\mathcal{F}$ on M is bimeromorphically equivalent to $\mathcal{G}$ on N if $\mathcal{F}_{|M-\Sigma} = (\phi_{|M-\Sigma})^*(\mathcal{G}_{|N-S})$, which is denoted by $\mathcal{F} = \phi^*(\mathcal{G})$.

Our next definitions are based on two facts about foliations on surfaces. The first fact is that the holomorphic tangent field of a singular foliation $\mathcal{F}$ along its regular part has a unique extension to an abstract holomorphic line bundle over M (cf. [GM]), which is denoted by $T_{\mathcal{F}}$. The second fact is Seidenberg's theorem (cf. [S]), which asserts that after a finite number of blow ups the singularities of foliations become *reduced*, i.e. locally generated by holomorphic vector fields whose linear part have eigenvalues 1 and $\lambda \notin \mathbf{Q}^+$. We shall say that $\mathcal{F}$ is reduced if all its singularities are reduced.

The *cotangent* line bundle is the dual line bundle of $T_{\mathcal{F}}$, denoted by $T^*_{\mathcal{F}}$. Consider now the meromorphic mapping $\phi_n : M- \to \mathbb{CP}^r$ given by $\phi_n(x) := (s_0(x) : \ldots : s_{r+1}(x))$, where $s_0, \ldots, s_{r+1}$ is a $\mathbf{C}$-basis of $H^0(M, T_{\mathcal{F}}^{*\otimes n})$. The image of M by ϕ_n is defined as the closure $\phi_n(M) := \overline{\phi_n(M \setminus I)}$ in $\mathbb{CP}^r$, where $I \subset M$ is indeterminacy locus of ϕ_n. We define the *cotangent dimension* as

$$\kappa(M, T^*_{\mathcal{F}}) := \max_{n \in \mathbf{N}} \{dim_{\mathbf{C}}\, \phi_n(M)\}$$

or

$$\kappa(M, T^*_{\mathcal{F}}) := -\infty \quad \text{if} \quad H^0(M, T_{\mathcal{F}}^{*\otimes n}) = 0 \quad \forall n \geq 1.$$

If $\widetilde{\mathcal{F}}$ on $\widetilde{M}$ is any reduced foliation bimeromorphically equivalent to $\mathcal{F}$, then the *Kodaira dimension* of $\mathcal{F}$ is defined as $\kappa(\mathcal{F}) := \kappa(\widetilde{M}, T^*_{\widetilde{\mathcal{F}}})$.

In the proof of Theorem 3.1.1 we will see that the definition of $\kappa(\mathcal{F})$ is coherent and that $\kappa(\mathcal{F})$ is a bimeromorphic invariant of $\mathcal{F}$. Let us state some results concerning foliations on projective surfaces. We shall say that a foliation $\mathcal{F}$ on M is *deformable* if there exists another foliation $\mathcal{G}$ on M such that $T^*_{\mathcal{G}}$ and $T^*_{\mathcal{F}}$ are isomorphic line bundles. If $\kappa(M)$ denotes the *Kodaira dimension* of M (cf. § 2.2) and $\mathcal{F}$ is a deformable foliation on M, then $\kappa(M) \leq \kappa(M, T^*_{\mathcal{F}})$ (Proposition 3.2.1). We prove that $\kappa(M, T^*_{\mathcal{F}}) \leq 1$ if some leaf of $\mathcal{F}$ is a generic fiber of an elliptic fibration or if $\mathcal{F}$ is transverse to a generic fiber of a fibration; conversely, if $\kappa(M, T^*_{\mathcal{F}}) = 1$ then either $\mathcal{F}$ is an elliptic fibration or $\mathcal{F}$ is generically transverse to a fibration (Theorem 3.3.1). If $\kappa(M, T^*_{\mathcal{F}}) = -\infty$, then either $\mathcal{F}$ is birationally equivalent to a rational fibration or $\mathcal{F}$ is *not* deformable. If $\kappa(M, T^*_{\mathcal{F}}) = -\infty$ and M is a rational surface with non-negative *anti-Kodaira* dimension (cf. § 3.4), then $\mathcal{F}$ is birationally equivalent to a rational fibration (Theorem 3.4.1).

The dimension $\kappa(\mathcal{F})$ is the coarsest bimeromorphic invariant and the classification of foliations can be refined. In the last section (§ 3.5) we introduce another invariant, which is the foliated analogue of the geometric genus of curves on surfaces.

2 Preliminaries

2.1

We present here some facts on foliations (cf. [Br2]), for the reader's convenience. A holomorphic singular foliation $\mathcal{F}$ on a compact complex surface M is determined by a holomorphic bundle map $\alpha : T_{\mathcal{F}} \to TM$ with isolated zeros (defined up to constant), where $T_{\mathcal{F}}$ is the extended tangent line bundle of $\mathcal{F}$ and TM is the holomorphic tangent bundle of M. The bundle map α associates to a holomorphic (meromorphic) section s of $T_{\mathcal{F}}$ a holomorphic (meromorphic) vector field $\mathcal{X} := \alpha \circ s$ inducing the foliation. On projective surfaces we can define a global meromorphic vector field $\mathcal{X}$ inducing $\mathcal{F}$ and $T_{\mathcal{F}} = \mathcal{O}_M((\mathcal{X})_0 - (\mathcal{X})_\infty)$, if $(\mathcal{X})_0$ and $(\mathcal{X})_\infty$ denote the divisors of zeroes and poles, resp.

The *normal* line bundle $N_{\mathcal{F}}$ can be defined as follows. Let $\{\mathcal{U}_i\}$ be an open covering of M in which $\mathcal{F}_{|\mathcal{U}_i}$ is given by $\omega_i = 0$, where ω_i are holomorphic 1-forms with isolated singularities. We have $\omega_i = h_{ij}\omega_j$ in $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$ and $N_{\mathcal{F}}$ is defined (up to isomorphism) by the $\mathcal{O}^*$-cocycle $\{h_{ij}\}$. The *canonical* line bundle of the surface M is defined as $K_M := \bigwedge^2 TM^*$ and the contraction operation gives the isomorphism $K_M \simeq Hom(T_{\mathcal{F}}, N^*_{\mathcal{F}})$, i.e.

$$K_M \simeq T^*_{\mathcal{F}} \otimes N^*_{\mathcal{F}}. \tag{1}$$

The *divisor of tangencies* $D_{\mathcal{F}\mathcal{G}} = \sum_i k_i C_i$ of a pair of foliations $\mathcal{F}$ and $\mathcal{G}$ on M has as *support* the set $\cup_i C_i$ of points where $\mathcal{F}$ and $\mathcal{G}$ are not transverse and $k_i \geq 1$ is the order of tangency at a generic point of the irreducible curve C_i. From the definitions, we have

$$N^*_{\mathcal{F}} \otimes T_{\mathcal{G}} \simeq \mathcal{O}_M(-D_{\mathcal{F}\mathcal{G}}).$$

Let us suppose to what follows that a foliation $\mathcal{F}$ on M is locally induced at $p = (0, 0)$ by a holomorphic vector field

$$\mathcal{X} = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y}.$$

Let $C = \sum_{i=1}^r C_i$ be a reduced curve of M with local equation $f_p = 0$ at p.

Suppose that all local components of C at p are *not* $\mathcal{F}$-invariant. Consider the ideal $\mathcal{I} = \langle f_p, \mathcal{X}(f_p)\rangle$ and define $tang(\mathcal{F}, C, p) := dim_{\mathbb{C}}\mathcal{O}_{\mathbb{C}^2}/\mathcal{I}$. If all components C_i of C are not $\mathcal{F}$-invariant, we define $tang(\mathcal{F}, C) = \sum_{p\in C} tang(\mathcal{F}, C, p)$ and obtain:

$$T^*_{\mathcal{F}} \cdot C = tang(\mathcal{F}, C) - C^2. \tag{2}$$

If we define the *degree* of a foliation on the projective plane $\mathbb{CP}^2$ as $d(\mathcal{F}) := tang(\mathcal{F}, L)$, where L is any (non $\mathcal{F}$-invariant) straight line, then (2) gives $T^*_{\mathcal{F}} = \mathcal{O}_{\mathbb{CP}^2}(d(\mathcal{F})-1)$ and thus $\kappa(\mathbb{CP}^2, T^*_{\mathcal{F}}) = -\infty, 0, 2$ according if $d(\mathcal{F}) = 0, 1, \geq 2$, resp.

Suppose now that C is a $\mathcal{F}$-invariant smooth (connected) curve. Denote by $Z(\mathcal{F}, C, p)$ the vanishing order of $\mathcal{X}_{|C}$ at p and define

$$Z(\mathcal{F}, C) := \sum_{p\in C} Z(\mathcal{F}, C, p).$$

Then we obtain:

$$T^*_{\mathcal{F}} \cdot C = Z(\mathcal{F}, C) - \chi(C), \tag{3}$$

where $\chi(C)$ is the Euler characteristic (this fact can be extended to singular reduced curves, by means of the GSV index [G-S-V] and $\chi(C) := -K_M \cdot C - C^2$).

Consider the ideal $\mathcal{J} =< a(x, y), b(x, y) >$, for $a(x, y), b(x, y)$ the local components of $\mathcal{X}$ and put

$$Det(\mathcal{F}, p) := dim_{\mathbb{C}}\mathcal{O}_p/\mathcal{J}, \quad Det(\mathcal{F}) := \sum_{p\in M} Det(\mathcal{F}, p).$$

According to Baum-Bott's formula (see e.g. [Br2]):

$$Det(\mathcal{F}) = c_2(M) + T^*_{\mathcal{F}} \cdot (T^*_{\mathcal{F}} \otimes K^*_M), \tag{4}$$

where $c_2(M)$ is the Euler number of M.

2.2

We present here facts on D-dimension (cf. [I1], [I2]) which will be used along the paper.

Let D be any holomorphic line bundle over a compact complex surface M. Consider the meromorphic mapping $\phi_n(D) : M- \to \mathbb{CP}^r$ given by $\phi_n(x) := (s_0(x) : \ldots : s_{r+1}(x))$, where $s_0, \ldots, s_{r+1}$ is a $\mathbf{C}$-basis of $H^0(M, D^{\otimes n})$. The *D-dimension* is defined as

$$\kappa(M, D) := \max_{n \in \mathbf{N}} \{dim_{\mathbf{C}}\, \phi_n(D)(M)\}$$

or

$$\kappa(M, D) := -\infty \quad \text{if} \quad H^0(M, D^{\otimes n}) = 0 \quad \forall n \geq 1.$$

From the definition it follows that $\kappa(M, D) \leq dim_{\mathbf{C}}(M)$ and that $\kappa(M, D) = dim_{\mathbf{C}}(M)$ if D is ample. It is not difficult to prove that a): $\kappa(M, D) = \kappa(M, D^{\otimes n})$, for any $n \geq 1$ and that b): $\kappa(M, D_1) \leq \kappa(M, D_1 \otimes D_2)$, if $\kappa(M, D_2) \geq 0$.

A useful characterization of D-dimension is the following: there exist $n_0 \in \mathbf{N}$ and $\alpha, \beta > 0$ such that for $n >> 1$:

$$\alpha n^{\kappa(M,D)} \leq h^0(D^{\otimes n_0 n}) \leq \beta n^{\kappa(M,D)},$$

where $h^0(\cdot) := dim_{\mathbf{C}} H^0(M, \cdot)$ and $n^{-\infty} := 0$.

The *Kodaira dimension* of a n-dimensional variety M is defined as $\kappa(M) := \kappa(M, K_M)$, where $K_M := \bigwedge^n TM^*$ is the *canonical* line bundle. If M is a compact Riemann surface, then $\kappa(M) = -\infty, 0, 1$ according if the genus verifies $g(M) = 0, 1, \geq 2$, resp. We refer [B-P-V] and [R] for the role of $\kappa(M)$ in the bimeromorphic classification of surfaces.

3 Proofs

3.1

Theorem 3.1.1 *Let $\mathcal{F}$ and $\mathcal{G}$ be holomorphic singular foliations on compact complex surfaces. If $\mathcal{F}$ and $\mathcal{G}$ are bimeromorphically equivalent, then $\kappa(\mathcal{F}) = \kappa(\mathcal{G})$.*

Proof: The proof follows immediately from the definitions and the following fact: If $\mathcal{F}_1$ and $\mathcal{F}_2$ are bimeromorphically equivalent *reduced* foliations on M_1 and M_2, resp., then

$$h^0(T_{\mathcal{F}_1}^{*\otimes n}) = h^0(M_2, T_{\mathcal{F}_2}^{*\otimes n}) \quad \forall n \geq 1.$$

In order to prove this assertion, let $\phi : M_1 - \to M_2$ be a bimeromorphic transformation such that $\mathcal{F}_1 = \phi^*(\mathcal{F}_2)$. By the theorem of elimination of indeterminations [B-P-V], there exist a compact complex surface S and sequences of blow ups (and isomorphisms) $\Sigma_i : S \to M_i$ such that $\Sigma_1^*(\mathcal{F}_1) = \Sigma_2^*(\mathcal{F}_2)$. Let us show that, for $\mathcal{G} := \Sigma_i^*(\mathcal{F}_i)$, $i = 1, 2$:

$$H^0(S, T_{\mathcal{G}}^{*\otimes n}) = H^0(M_i, T_{\mathcal{F}_i}^{*\otimes n}), \quad \forall n \geq 1.$$

Without loosing generality, suppose that Σ_i is just one blow-up σ_i of a point $p_i \in M_i$, with $E_i = \sigma_i^{-1}(p_i)$. If ω_i is a local 1-form inducing $\mathcal{F}_i$ at p_i and $m_{p_i} \geq 0$ is the vanishing order of $\sigma_i^*(\omega_i)$ along E_i, then $N_{\mathcal{G}}^* = \sigma_i^*(N_{\mathcal{F}_i}^*) \otimes \mathcal{O}_S(m_{p_i} E_i)$ (cf. § 2.1). Using the isomorphisms $K_S = \sigma_i^*(K_{M_i}) \otimes \mathcal{O}_S(E_i)$ and $T_{\mathcal{G}}^* = N_{\mathcal{G}} \otimes K_S$, we obtain:

$$T_{\mathcal{G}}^* = \sigma_i^*(T_{\mathcal{F}_i}^*) \otimes \mathcal{O}_S((1 - m_{p_i})E_i). \tag{5}$$

Since $\mathcal{F}_i$ are reduced foliations, $m_{p_i} \in \{0, 1\}$. In the case $m_{p_i} = 1$, the result follows from the fact that $H^0(S, \sigma_i^*(L)) = H^0(M_i, L)$, for any line bundle L (see e.g. [B-P-V], [I2]). In the case $m_{p_i} = 0$ the result follows from a slightly more general fact:

$$H^0(S, \sigma_i^*(L) \otimes \mathcal{O}_S(nE_i)) = H^0(M_i, L), \quad n \geq 1,$$

which can be reduced to previous fact as follows. Define $L_n := \sigma_i^*(L) \otimes \mathcal{O}_S(nE_i)$ and consider the exact sequence of restriction: $0 \to L_{n-1} := L_n \otimes \mathcal{O}_S(-E) \to L_n \to L_{n\,|E} \to 0$. Using the property $\sigma_i^*(L) \cdot E_i = 0$ we obtain $L_{n\,|E_i} = \mathcal{O}_S(nE_i)_{|E_i}$ and hence $L_{n\,|E_i} \simeq \mathcal{O}_{\mathbb{CP}^1}(-n)$. From the exact sequence $0 \to L_{n-1} \to L_n \to \mathcal{O}_{\mathbb{CP}^1}(-n) \to 0$ we obtain, passing to the exact sequence in cohomology, $H^0(S, L_{n-1}) \simeq H^0(S, L_n)$. □

3.2

We say that a holomorphic singular foliation $\mathcal{F}$ on M is *deformable* if there exists another foliation $\mathcal{G}$ on M such that $T_{\mathcal{G}}^*$ and $T_{\mathcal{F}}^*$ are isomorphic holomorphic line bundles. In Remarks 3.3.2 and 3.4.2 we give examples of non-deformable foliations.

Proposition 3.2.1 *If $\mathcal{F}$ is a deformable foliation on a projective surface, then $\kappa(M) \leq \kappa(M, T_{\mathcal{F}}^*)$.*

Proof : If $D_{\mathcal{F}\mathcal{G}}$ denotes the divisor of tangencies of $\mathcal{F}$ and $\mathcal{G}$, then

$$N_{\mathcal{F}} \otimes T^*_{\mathcal{G}} \simeq \mathcal{O}_M(D_{\mathcal{F}\mathcal{G}}),$$

cf. § 2.1. If $\mathcal{F}$ is a deformable foliation and $T^*_{\mathcal{F}} \simeq T^*_{\mathcal{G}}$, then $\mathcal{O}_M(D_{\mathcal{F}\mathcal{G}}) \simeq N_{\mathcal{F}} \otimes T^*_{\mathcal{F}}$; from $T^*_{\mathcal{F}} \simeq K_M \otimes N_{\mathcal{F}}$ we obtain that $T^{*\otimes 2}_{\mathcal{F}} \simeq \mathcal{O}_M(D_{\mathcal{F}\mathcal{G}}) \otimes K_M$. Since $D_{\mathcal{F}\mathcal{G}}$ is an effective divisor, we obtain from properties a), b) of D-dimension in § 2.2 that $\kappa(M, T^*_{\mathcal{F}}) \geq \kappa(M, K_M)$. □

3.3

A *fibration* $\mathcal{P}$ of a surface M over a compact Riemann surface B means a surjective holomorphic mapping $p : M \to B$. We regard $\mathcal{P}$ as a singular foliation with holomorphic first integral. A *generic* (resp. *critical*) fiber of $\mathcal{P}$ is the pre-image of a regular (resp. critical) value of $p : M \to B$. We shall say that a fibration is *connected* if its generic fiber is connected. A fibration is said to be *rational* (resp. *elliptic*) if its generic fiber is the Riemann sphere $\mathbb{CP}^1$ (resp. an elliptic curve).

The basic examples of singular foliations transverse to a fibration $\mathcal{P}$ along generic fibers are given by *Riccati* foliations, when $\mathcal{P}$ is rational, and *turbulent* foliations, when $\mathcal{P}$ is elliptic (see [Br2]). In both cases, a finite number of fibers C_i are $\mathcal{F}$-invariant and, for non-trivial Riccati foliations, $Sing\mathcal{F} \cap C_i \neq \emptyset$.

Next result is the foliated version of the Fibering theorem of [I1]:

Theorem 3.3.1 *Let $\mathcal{F}$ be a foliation of a projective surface M.*

1) *If $\kappa(M, T^*_{\mathcal{F}}) = 1$, then there exists a fibration $\mathcal{P}$ of M; moreover, either $\mathcal{F}$ is transverse to the generic fiber of $\mathcal{P}$ or $\mathcal{F} = \mathcal{P}$ is an elliptic fibration.*

2) *Suppose that there exists a connected fibration $\mathcal{P}$ of M, with generic fiber C. If C is $\mathcal{F}$-invariant and $Sing\mathcal{F} \cap C = \emptyset$, then $\kappa(M, T^*_{\mathcal{F}}) \leq \kappa(C) + 1$. If $\mathcal{F}$ is transverse to C, then $\kappa(M, T^*_{\mathcal{F}}) \leq 1$.*

Proof :

Item 1): Since $\kappa(M, T^*_{\mathcal{F}}) = 1$, there exist an integer $n_0 \geq 1$ and sections $s_1, s_2 \in H^0(M, T^{*\otimes n_0}_{\mathcal{F}})$ such that $\phi := \frac{s_1}{s_2} : M - \to B$ is a meromorphic mapping onto a (singular) complex curve B (for notation brevity, let $n_0 = 1$).

After normalization $r : B' \to B$ we obtain a compact Riemann surface B' and a meromorphic mapping $\phi' : M - \to B'$ such that $\phi = r \circ \phi'$. We assert that ϕ' is well-defined on all M, i.e. there exists a fibration $\mathcal{P}$ given by $\phi' : M \to B'$. In order to prove this assertion, let us suppose by absurd that $I := (s_1)_0 \cap (s_2)_0 \neq \emptyset$, where $(s_1)_0$, $(s_2)_0$ denote the curves of zeroes. After passing to the mobile part of the linear system $|T^*_{\mathcal{F}}|$ we can suppose that $(s_1)_0$ and $(s_2)_0$ have no common component. Then $I \neq \emptyset$ gives $(s_1)_0 \cdot (s_2)_0 > 0$ and hence $(T^*_{\mathcal{F}})^2 = (s_1)_0 \cdot (s_2)_0 > 0$. Let us show that this condition implies $\kappa(M, T^*_{\mathcal{F}}) = 2$, a contradiction. Let A be any ample line bundle of M. Since $T^*_{\mathcal{F}}$ has global holomorphic sections, $h^0(K_M \otimes A) \geq h^0(K_M \otimes A \otimes T_{\mathcal{F}}^{\otimes n})$, for $n \geq 1$. By Serre's duality, $h^2(T_{\mathcal{F}}^{*\otimes n} \otimes A^*) = h^0(K_M \otimes A \otimes T_{\mathcal{F}}^{\otimes n})$ and applying Riemann-Roch's theorem to $T_{\mathcal{F}}^{*\otimes n} \otimes A^*$ we obtain:

$$h^0(T_{\mathcal{F}}^{*\otimes n} \otimes A^*) + h^0(K_M \otimes A) \geq \chi(\mathcal{O}_M) + \frac{n^2}{2}(T^*_{\mathcal{F}})^2 + r(n),$$

where $r(n)$ is a linear function of n, and $(T^*_{\mathcal{F}})^2 > 0$ implies that $h^0(T_{\mathcal{F}}^{*\otimes n} \otimes A^*) > 0$ for $n >> 1$. Using properties a), b) of D-dimension in § 2.2, we obtain $\kappa(M, T^*_{\mathcal{F}}) = \kappa(M, T_{\mathcal{F}}^{*\otimes n} \otimes A^* \otimes A) \geq \kappa(M, A) = 2$.

Let $C = \phi'^{-1}(x)$ be a generic fiber of the fibration $\mathcal{P}$: $\phi' : M \to B'$. By the previous reasoning, we obtain that $T^*_{\mathcal{F}} \cdot C = 0$. If C is $\mathcal{F}$-invariant, then $\mathcal{F} = \mathcal{P}$ and from (3) in §2.1 we obtain: $T^*_{\mathcal{P}} \cdot C = -\chi(C) = 0$, i.e. C is an elliptic curve and $\mathcal{P}$ is an elliptic fibration. If C is supposed not $\mathcal{F}$-invariant, we obtain from (2) in §2.1: $T^*_{\mathcal{F}} \cdot C = tang(\mathcal{F}, C) = 0$, i.e. transversality between $\mathcal{F}$ and $\mathcal{P}$ along C.

Item 2): Suppose that $\mathcal{P}$ is given as $p : M \to B$, where B is a compact Riemann surface. According to an Addition Formula [I2],

$$\kappa(M, D) \leq \kappa(p^{-1}(x), D_{|p^{-1}(x)}) + 1,$$

for any holomorphic line bundle D over M and any regular value $x \in B$. Let us apply this fact to $D = T^*_{\mathcal{F}}$. If a generic fiber $C = p^{-1}(x)$ is $\mathcal{F}$-invariant and $Sing\mathcal{F} \cap C = \emptyset$, then $(T^*_{\mathcal{F}})_{|C} \simeq T^*_C = K_C$ and thus $\kappa(M, T^*_{\mathcal{F}}) \leq \kappa(C) + 1$. If we suppose now that $\mathcal{F}$ is transverse to a generic fiber C, then $(T^*_{\mathcal{F}})_{|C} \simeq \mathcal{O}_C$ and $\kappa(M, T^*_{\mathcal{F}}) \leq 1$. □

Remark 3.3.2 An immediate corollary of item 2) of Theorem 3.3.1 combined with Proposition 3.2.1: if $\mathcal{P}$ is a fibration of projective surface M with $\kappa(M) = 2$ and $\mathcal{F}$ is a foliation transverse to the generic fiber of $\mathcal{P}$, then $\mathcal{F}$ is not deformable. For example, if M is the product of compact Riemann surfaces B and C with $g(B), g(C) \geq 2$, then the horizontal and vertical fibrations are not deformable foliations.

Remark 3.3.3 We give examples of the equality $\kappa(M, T^*_{\mathcal{P}}) = \kappa(C) + 1 = 2$, for a connected fibration $\mathcal{P}$ with generic fiber C. First, we assert that:

$$T^*_{\mathcal{P}} = K_M \otimes p^*(K^*_B) \otimes \mathcal{O}_M\left(\sum_{s,i}(1 - n_{si})C_{si}\right), \tag{6}$$

where $C_s = \sum_i n_{si} C_{si}$ is the decomposition of a critical fiber C_s in irreducible components. In fact, since $T^*_{\mathcal{P}} = K_M \otimes N_{\mathcal{P}}$, we need to show that $N_{\mathcal{P}} = p^*(K^*_B) \otimes \mathcal{O}_M(\sum_{s,i}(1 - n_{si})C_{si})$. In order to show this, consider $\eta = f(z)dz$ a local non-singular holomorphic 1-form of B. Since $\mathcal{P}$ is connected, $p^*(\eta)$ has at most isolated singularities along $M - \cup_s C_s$ and $p^*(\eta)$ has zeroes of order $n_{si} - 1$ along the regular part of C_{si}. Thus $p^*(\eta) = 0$ induces $\mathcal{P}$ as a foliation on $M - \cup_s C_s$ and we obtain: $p^*(K_B) = N^*_{\mathcal{P}} \otimes \mathcal{O}_M(\sum_{s,i}(1 - n_{si})C_{si})$, as desired.

Therefore, $T^*_{\mathcal{P}} = K_M \otimes p^*(K^*_B)$ holds for $\mathcal{P}$ having only reduced fibers (i.e. free of multiplicity). The line bundle $\omega_{M/B} := K_M \otimes p^*(K^*_B)$ is called the *relative* canonical bundle or the *dualising sheaf.* If $\mathcal{P}$ is a semi-stable*, non-isotrivial* connected fibration, free from (-2)-curves contained in its fibers and such that its generic fiber C has genus $g(C) \geq 2$, then $\omega_{M/B}$ is ample (cf. [Sz]); consequently $\kappa(\mathcal{P}) = \kappa(C) + 1 = 2$.

Remark 3.3.4 Let $\mathcal{F}$ be a reduced foliation on a projective surface M and suppose that there exists an entire map $f : \mathbf{C} \to M$ tangent to $\mathcal{F}$ with Zariski-dense image on M. We assert that $\kappa(M, T^*_{\mathcal{F}}) \leq 1$. In fact, according to [Mc] there exists a positive closed current ϕ associated to $f : \mathbf{C} \to M$, such that the cohomology class $[\phi] \in H^2(M, \mathbf{R})$ verifies

a) $[\phi]$ intersects non-negatively any curve and

b) $T^*_{\mathcal{F}} \cdot [\phi] \leq 0$ (see also [Br3]).

*A fibration is called *isotrivial* if the analytic structure of the generic fibers is fixed. A fibration is *semi-stable* if there is no exceptional lines (i.e. (-1)-curves) contained in its fibers and all critical fibers are reduced curves with at most nodal type singularities.

If we suppose by absurd that $\kappa(M, T^*_{\mathcal{F}}) = 2$, then $T^*_{\mathcal{F}}$ is ample out of a union of contractible curves and we get $T^*_{\mathcal{F}} \cdot [\phi] > 0$.

An immediate consequence on the projective plane: if $\mathcal{F}$ is a reduced foliation on $\mathbb{CP}^2$ and there exists an entire map $f : \mathbf{C} \to M$ tangent to $\mathcal{F}$ with Zariski-dense image, then $d(\mathcal{F}) \leq 1$. Very similar to the situation in the projective plane, a Riccati foliation on $\mathbb{CP}^1 \times \mathbb{CP}^1$ with $\kappa(\mathbb{CP}^1 \times \mathbb{CP}^1, T^*_{\mathcal{F}}) = 0$ can be represented in affine coordinates by $xdy - \lambda ydx = 0$, $\lambda \in \mathbf{C}$, and the entire curve $f(t) = (e^t, e^{\lambda t})$ is tangent to $\mathcal{F}$. Consider now an *Abelian* surface M, i.e. a projective torus, with a linear foliation $\mathcal{H}$ induced by a holomorphic 1-form $w = \lambda dx + \mu dy$ on $\mathbf{C}^2$, $\lambda, \mu \in \mathbf{C}$. It is easy to see that $N^*_{\mathcal{H}} = \mathcal{O}_M$ and $T^*_{\mathcal{H}} = \mathcal{O}_M$, i.e. $\kappa(M, T^*_{\mathcal{H}}) = 0$. For generic values λ, μ the leaves of $\mathcal{H}$ are transcendent and isomorphic to $\mathbf{C}$. There are also examples of turbulent foliations $\mathcal{G}$ with $\kappa(M, T^*_{\mathcal{G}}) \in \{0, 1\}$, for which all generic leaves are transcendent and isomorphic to $\mathbf{C}$ (a construction is due to A. Lins Neto).

3.4

We recall that a projective surface M has Kodaira dimension $\kappa(M) = -\infty$ if and only if M is *ruled*, i.e. birationally equivalent to $\mathbb{CP}^1 \times B$, where B is a compact Riemann surface (M is called *rational* if $B \simeq \mathbb{CP}^1$).

The *anti-Kodaira dimension* is defined as the dimension $\kappa(M, K^*_M)$ (cf. § 2.2). For rational surfaces, $\kappa(M, K^*_M) \geq 0$ holds for instance if M is

a) a *Del Pezzo* surface, i.e. $\mathbb{CP}^2$ blown-up in at most 8 points in general position,

b) a *Hirzebruch* surface Σ_n, i.e. a holomorphic $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$ having a holomorphic section σ_n with self-intersection $(\sigma_n)^2 = -n$, or

c) a Hirzebruch surface Σ_n blown-up in at most 7 points, if $n \leq 3$, or at most $n + 4$ points if $n \geq 4$ (cf. [Sa1]).

Theorem 3.4.1 *If $\mathcal{F}$ is a foliation on a projective surface M with $\kappa(M, T^*_{\mathcal{F}}) = -\infty$, then $\mathcal{F}$ is birationally equivalent to a rational fibration or $\mathcal{F}$ is not deformable. If additionally M is supposed to be a rational surface with $\kappa(M, K^*_M) \geq 0$, then $\mathcal{F}$ is birationally equivalent to a rational fibration.*

Remark 3.4.2 If $\mathcal{P}$ is a rational fibration, then $\kappa(M, T^*_{\mathcal{P}}) = -\infty$ and $\mathcal{P}$ is not deformable. In fact, suppose by absurd that there is a non-trivial section $s \in H^0(M, T^{*\otimes l}_{\mathcal{P}})$, for $l \geq 1$, and denote $(s)_0$ the divisor of zeroes. If F is a generic fiber of $\mathcal{P}$, then $(s)_0 \cdot F \geq 0$; but from (3) §2.1 we obtain $(s)_0 \cdot F = T_{\mathcal{P}}^{*\otimes l} \cdot F = -2l$, a contradiction. Suppose now by absurd that there exists $\mathcal{G} \neq \mathcal{F}$ with $T_{\mathcal{G}}^* \simeq T_{\mathcal{P}}^*$. Then a generic fiber F of $\mathcal{P}$ is not $\mathcal{G}$-invariant and from § 2.1 we obtain $T_{\mathcal{G}}^* \cdot F = tang(\mathcal{G}, F) \geq 0$; but also we have $T_{\mathcal{G}}^* \cdot F = T_{\mathcal{P}}^* \cdot F = -2$, contradiction.

Next fact (cf. [Mc]) is fundamental for the proof of Theorem 3.4.1:

Lemma 3.4.3 (Miyaoka's Semipositivity Theorem.) *Let $\mathcal{F}$ be a foliation on a projective surface M. If $\mathcal{F}$ is not birationally equivalent to a rational rational fibration, then $T_{\mathcal{F}}^*$ is pseudo-effective, i.e. $T_{\mathcal{F}}^*$ intersects non-negatively any ample divisor of M.*

For generalizations of the next lemma we refer [Sa2]:

Lemma 3.4.4 *Let D be a pseudo-effective divisor* of a rational surface M. If $K_M + D$ is pseudo-effective, then $\kappa(M, K_M + D) \geq 0$.*

Proof (of Lemma 3.4.4.) The Zariski decomposition (cf. [F]) asserts that any pseudo-effective divisor L has a unique decomposition $L = P + N$ such that

i) $N = \sum_i q_i N_i$ is an effective $\mathbf{Q}$-divisor and either $N = 0$ or the quadratic form represented by the matrix $(N_i \cdot N_j)$ is negative definite and

ii) P is a nef $\mathbf{Q}$-divisor with $P \cdot N_i = 0, \forall i$.

Consider now the Zariski decomposition $K_M + D = P + N$. Since N is an effective $\mathbf{Q}$-divisor, in order to obtain $\kappa(M, K_M + D) \geq 0$ it is enough to prove that $h^0(nP) > 0$ for $n >> 1$ (and such that nP and nN are divisors). On a rational surface we can suppose that $-P$ is not pseudo-effective: otherwise we show that P is numerically trivial and $P = 0$, i.e. $h^0(P) > 0$, as desired. We assert that $h^0(K_M - nP) = 0$ for $n >> 1$. In fact, let A be an ample divisor such that $-P \cdot A < 0$. If $h^0(K_M - nP) > 0$ for a sequence $n \to +\infty$, then $E_n := \frac{1}{n} K_M - P$ is an effective $\mathbf{Q}$-divisor and from $A \cdot E_n > 0$ we conclude that $-P \cdot A \geq 0$, a contradiction.

Since $\chi(\mathcal{O}_M) = 1$ and $h^2(nP) = h^0(K_M - nP)$ (Serre's duality), we obtain from Riemann-Roch's theorem: $h^0(nP) \geq 1 + \frac{1}{2}(n^2 P^2 - nP \cdot K_M)$, if $n >> 1$.

*Along this section we use the correspondence between line bundles and divisors on projective manifolds. By a *$\mathbf{Q}$-divisor* $D = \sum_i q_i C_i$ we mean a finite formal sum of irreducible curves C_i with rational coefficients; D is *effective* if $q_i \geq 0\ \forall i$. A divisor ($\mathbf{Q}$-divisor) D is *pseudo-effective* (resp. *nef*) if $D \cdot C \geq 0$ for all ample divisors C (resp. all curves C) of M.

A fundamental fact on nef divisors is the inequality $(nP)^2 \geq 0$ (cf. [R]). In the case $P^2 > 0$, we obtain $h^0(nP) > 0$ for $n >> 1$, as desired. Therefore we can suppose that $P^2 = 0$ and it is enough to show that $P \cdot K_M \leq 0$. After intersecting with P we obtain $P \cdot (K_M + D) = P \cdot (P + N) = 0$, since $P \cdot N = 0$, i.e. $D \cdot P = -P \cdot K_M$. Roughly speaking, nef divisors are limit of ample divisors and therefore $D \cdot P \geq 0$, since nP is nef and D is pseudo-effective (we refer [R] for a proof of this fact); the proof is completed. □

Proof (of Theorem 3.4.1.) Supposing that $\kappa(M, T^*_{\mathcal{F}}) = -\infty$ and that $\mathcal{F}$ is deformable, let us show that $\mathcal{F}$ is birationally equivalent to a rational fibration. Suppose by absurd that $\mathcal{F}$ is not birationally equivalent to a rational fibration. Then Miyaoka's Theorem (Lemma 3.4.3) implies that $T^{*\otimes 2}_{\mathcal{F}}$ is pseudo-effective. As in the proof of Proposition 3.2.1, we obtain the isomorphism $T^{*\otimes 2}_{\mathcal{F}} \simeq K_M \otimes \mathcal{O}_M(D_{\mathcal{F}\mathcal{G}})$, where $D_{\mathcal{F}\mathcal{G}}$ is the divisor of tangencies between $\mathcal{F}$ and $\mathcal{G} \neq \mathcal{F}$ with $T^*_{\mathcal{G}} \simeq T^*_{\mathcal{F}}$. Since $D_{\mathcal{F}\mathcal{G}}$ is an effective divisor, we obtain the contradiction $\kappa(M, T^*_{\mathcal{F}}) = \kappa(M, K_M \otimes \mathcal{O}_M(D_{\mathcal{F}\mathcal{G}})) \geq 0$, using Lemma 3.4.4, provided that M is a rational surface.

We assert that, since $\mathcal{F}$ is a deformable foliation and $h^0(T^{*\otimes 4}_{\mathcal{F}}) = 0$, then M must be a rational surface. Let us prove this assertion: if M is not rational, then either $h^0(K_M^{\otimes 2}) > 0$ or $h^1(\mathcal{O}_M) > 0$, according to Castelnuovo's criterion [B-P-V]. For $\mathcal{G} \neq \mathcal{F}$ with $T^*_{\mathcal{G}} \simeq T^*_{\mathcal{F}}$, we have $T^{*\otimes 2}_{\mathcal{F}} \simeq K_M \otimes \mathcal{O}(D_{\mathcal{F}\mathcal{G}})$. If $h^0(K_M^{\otimes 2}) > 0$, then $h^0(T^{*\otimes 4}_{\mathcal{F}}) = h^0(K_M^{\otimes 2} \otimes \mathcal{O}(D_{\mathcal{F}\mathcal{G}})^{\otimes 2}) > 0$, a contradiction. If $h^1(\mathcal{O}_M) > 0$, then $h^0(\Omega^1_M) = h^1(\mathcal{O}_M) > 0$, i.e. there exists a non-trivial global holomorphic 1-form Ω on M. Since $\mathcal{F} \neq \mathcal{G}$, then Ω induces a (non-trivial) global section either of $T^*_{\mathcal{F}}$ or of $T^*_{\mathcal{G}}$. But $T^*_{\mathcal{F}} \simeq T^*_{\mathcal{G}}$ and, from the hypothesis, $h^0(T^*_{\mathcal{F}}) = 0$, a contradiction; hence M is rational.

For proving the second part of the theorem, let us suppose that $\kappa(M, T^*_{\mathcal{F}}) = -\infty$ and that M is a rational surface with $\kappa(M, K^*_M) \geq 0$. If by absurd we suppose that $\mathcal{F}$ is not birationally equivalent to a rational fibration, then $T^*_{\mathcal{F}}$ is pseudo-effective, according to Lemma 3.4.3. Since $\kappa(M, K^*_M) \geq 0$, then K^*_M is pseudo-effective and we conclude that $N_{\mathcal{F}} \simeq T^*_{\mathcal{F}} \otimes K^*_M$ is pseudo-effective. Using Lemma 3.4.4 we obtain $\kappa(M, T^*_{\mathcal{F}}) = \kappa(M, K_M \otimes N_{\mathcal{F}}) \geq 0$, a contradiction. □

3.5

The aim of this section is to introduce another bimeromorphic invariant of foliations, which is the foliated analogue of the geometric genus of curves on surfaces.

If $\mathcal{F}$ is a foliation on a compact complex surface M and $\widetilde{\mathcal{F}}$ on $\widetilde{M}$ is a reduced foliation bimeromorphically equivalent to $\mathcal{F}$, we define

$$g(\mathcal{F}) := \chi(\mathcal{O}_M) + \frac{1}{2} T^*_{\widetilde{\mathcal{F}}} \cdot (T^*_{\widetilde{\mathcal{F}}} \otimes K^*_{\widetilde{M}}).$$

We remark that if $\mathcal{G}$ on N is obtained from $\widetilde{\mathcal{F}}$ by a blow up σ at $p \in \widetilde{M}$, then

$$T^*_{\mathcal{G}} \cdot (T^*_{\mathcal{G}} \otimes K^*_N) = T^*_{\widetilde{\mathcal{F}}} \cdot (T^*_{\widetilde{\mathcal{F}}} \otimes K^*_{\widetilde{M}}) - m_p(m_p - 1), \tag{7}$$

where m_p is the vanishing order of $\sigma^*(\omega)$ along $\sigma^{-1}(p)$, for a 1-form ω inducing $\widetilde{\mathcal{F}}$ around p. Since the singularities of $\widetilde{\mathcal{F}}$ are reduced, then $m_p \in \{0, 1\}$ and we conclude that $T^*_{\mathcal{G}} \cdot (T^*_{\mathcal{G}} \otimes K^*_N) = T^*_{\widetilde{\mathcal{F}}} \cdot (T^*_{\widetilde{\mathcal{F}}} \otimes K^*_{\widetilde{M}})$. After this remark, using the fact that $\chi(\mathcal{O}_M)$ is a bimeromorphic invariant of surfaces [B-P-V] and adapting the proof of Theorem 3.1.1, we conclude that $g(\mathcal{F})$ does not depend on the particular reduced foliation $\widetilde{\mathcal{F}}$ and that $g(\mathcal{F})$ is a bimeromorphic invariant of $\mathcal{F}$.

The variation in (7) motivates the following definitions. Denoting by $\mathcal{R}(\mathcal{F})$ a reduction of singularities of $\mathcal{F}$, given as a sequence of blow-ups $\phi = \sigma_1 \circ \cdots \circ \sigma_n : \widetilde{M} \to M$ with $\widetilde{\mathcal{F}} = \phi^*(\mathcal{F})$, let us define the set $Sing\mathcal{R}(\mathcal{F}, p)$ as the union of the singularity $\{p\}$ and all singularites of $\sigma_1^*(\mathcal{F}), (\sigma_1 \circ \sigma_2)^*(\mathcal{F}), \ldots, \widetilde{\mathcal{F}}$ which belong to $\phi^{-1}(p)$, and put

$$\delta_p(\mathcal{F}) := \sum_{q \in Sing\mathcal{R}(\mathcal{F}, p)} \frac{m_q(m_q - 1)}{2}.$$

We obtain from (7): $\quad g(\mathcal{F}) = \chi(\mathcal{O}_M) + \frac{1}{2} T^*_{\mathcal{F}} \cdot (T^*_{\mathcal{F}} \otimes K^*_M) - \sum_p \delta_p(\mathcal{F}).$

Example 3.5.1. On the projective plane, $g(\mathcal{F}) = \frac{1}{2} d(\mathcal{F})(d(\mathcal{F}) + 1) - \sum_p \delta_p(\mathcal{F})$, since $\chi(\mathcal{O}_{\mathbb{CP}^2}) = 1$ and $T^*_{\mathcal{F}} = \mathcal{O}_{\mathbb{CP}^2}(d(\mathcal{F}) - 1)$, where $d(\mathcal{F})$ is the degree. Consider now the standard *quadratic* transformation $Q : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$, given as $Q(x_0 : x_1 : x_2) = (x_1 x_2 : x_0 x_2 : x_0 x_1)$, which is a biholomorphism when restricted to $\mathbb{CP}^2 - \Delta$, where $\Delta := \{x_0 x_1 x_2 = 0\}$. The transformation Q factorizes as a composition of three blow ups on the vertices of Δ followed by blow downs of (the strict transforms of) lines $\{x_i = 0\}$ to points $p_i \in \mathbb{CP}^2$, $i = 0, 1, 2$. We can verify that Q transforms smooth conics containing the three vertices of Δ into straight lines not passing by p_i. Let us suppose, for simplicity, that $\mathcal{F}$ is a reduced foliation and that Δ is chosen in generic position relatively to $\mathcal{F}$. Under these hypotheses, let us consider the transformed foliation $(Q^{-1})^*(\mathcal{F})$. Thanks to the properties of the tangency index $tang(\mathcal{F}, C, p)$ (cf. § 2.1), we easily verify that if C is a generic smooth conic containing the three vertices of Δ and L is its transform by Q, then $tang(\mathcal{G}, L) = tang(\mathcal{F}, C)$. By definition $d(\mathcal{G}) = tang(\mathcal{G}, L)$ and from the fact $tang(\mathcal{F}, C) = 2d(\mathcal{F}) + 2$, we obtain $d(\mathcal{G}) = 2d(\mathcal{F}) + 2$. The generic choice of Δ implies that the singularities p_i of $\mathcal{G}$, $i = 0, 1, 2$, do not give rise to other singularities under a blow up; from this we obtain $m_{p_i} = d(\mathcal{F}) + 2$ and $\delta_{p_i} = \frac{1}{2}(d(\mathcal{F}) + 2)(d(\mathcal{F}) + 1)$. Then $g(\mathcal{G}) = \frac{1}{2}(2d(\mathcal{F}) + 2)(2d(\mathcal{F}) + 3) - \sum_{i=0}^{2} \delta_{p_i} = g(\mathcal{F})$.

Theorem 3.5.2 *If $\mathcal{P}$ is a connected rational fibration of M over B, then $g(\mathcal{P}) = g(B) - 1$. If $\mathcal{P}$ is a connected elliptic fibration of M, then $g(\mathcal{P}) \leq \chi(\mathcal{O}_M)$.*

Proof : First, we assert that if $\mathcal{P}$ is any connected fibration of M over B, then:

$$\begin{aligned} 2\chi(\mathcal{O}_M) - 2g(\mathcal{P}) =& \chi(B)\chi(F) + \sum_s [\chi((F_s)_{red}) - \chi(F)] \\ & - Det(\mathcal{P}) + \sum_p 2\delta_p(\mathcal{P}), \end{aligned} \tag{8}$$

where F denotes a generic fiber, $(F_s)_{red}$ denotes the underlying reduced analytic set of a critical fiber F_s, $\chi((F_s)_{red})$ is the topological Euler characteristic and $Det(\mathcal{P})$ is the sum of indices appearing in Baum-Bott's formula (4) § 2.1.

In fact, $2\chi(\mathcal{O}_M) - 2g(\mathcal{P}) = -T^*_{\mathcal{P}} \cdot (T^*_{\mathcal{P}} \otimes K^*_M) + \sum_p 2\delta_p(\mathcal{P})$ and by Baum-Bott's formula:

$$2\chi(\mathcal{O}_M) - 2g(\mathcal{P}) = c_2(M) - Det(\mathcal{P}) + \sum_p 2\delta_p(\mathcal{P}), \tag{9}$$

where $c_2(M)$ is the Euler number of M. Since

$$c_2(M) = \chi(B)\chi(F) + \sum_s [\chi((F_s)_{red}) - \chi(F)]$$

(cf. [B-P-V]), the assertion is proved.

Let us suppose now that $\mathcal{P}$ is a rational fibration. It is known (cf. [B-P-V]) that there exist a sequence of blow downs $\Sigma : M \to M'$ and a regular rational fibration $\mathcal{P}'$ of M' over B such that $\mathcal{P} = \Sigma^*(\mathcal{P}')$. Since $\chi(\mathcal{O}_M) = \chi(\mathcal{O}_{M'}) = (1 - g(B))$ (cf. [B-P-V]) and $g(\mathcal{P}) = g(\mathcal{P}')$, the result follows immediately from remark (8) applied to $\mathcal{P}'$.

Suppose now that $\mathcal{P}$ is an elliptic fibration. Since $\chi(F) = 0$ and $\delta_p(\mathcal{P}) \geq 0$, (8) gives: $2\chi(\mathcal{O}_M) - 2g(\mathcal{P}) \geq \sum_s \chi((F_s)_{red}) - Det(\mathcal{P})$. Let us consider a restriction of $\mathcal{P}$ to an open set $\mathcal{U}_s$ containing exactly one critical fiber F_s; we want to show in this local setting that $\tau(s) := \chi((F_s)_{red}) - \sum_{p \in F_s} Det(\mathcal{P}, p) \geq 0$, for each s, which proves the proposition.

Case i): F_s does not contain exceptional lines. In this case, F_s is described in Kodaira's list (cf. [B-P-V]). Suppose that F_s is not a multiple fiber. By inspection of the list, we obtain (in the usual notation) $\tau(s) = 0$ if $F_s = I_{b \geq 0}, II, III, IV$, and $\tau(s) = 2$ if $F_s = I^*_{b \geq 0}, II^*, III^*, IV^*$ and we are done. If F_s is a multiple fiber, i.e. $F_s = k\tilde{F}$ for some $k \geq 2$, then $\tilde{F} = I_{b \geq 0}$ (cf. [B-P-V]); again we obtain $\tau(s) = 0$.

Case ii): F_s contains an exceptional line. In this case, F_s is obtained by blow ups from a critical fiber described in Case i). It is easy to verify that $\tau(s)$ is invariant by blow-ups in all cases considered in Case i), except for fibers of type III and IV. A blow up at the triple point of a fiber of type IV increases $\tau(s)$; a blow up σ at the singular point of a fiber of type III produces a triple point at $\sigma^{-1}(p)$, i.e. we obtain a fiber of type IV and we are done. □

There are examples of pencils of elliptic curves $\mathcal{F}$ on the projective plane with $g(\mathcal{F}) \in \{-2, -1, 0, 1\}$ (examples with $g(\mathcal{F}) = -2$ were constructed by A. Lins Neto), but we don't know if there exists a lower bound to $g(\mathcal{F})$. We can assert that any foliation on the plane with $g(\mathcal{F}) < 0$ has a singularity with an infinite number of local separatrices, what is proved in the next proposition:

Proposition 3.5.3 *Let $\mathcal{F}$ be a foliation on a compact complex surface M with $g(\mathcal{F}) < \chi(\mathcal{O}_M) - \frac{1}{2}c_2(M)$. Then there exists a singularity of $\mathcal{F}$ with an infinite number of local separatrices.*

Proof: Let $\widetilde{\mathcal{F}}$ be a reduced foliation obtained from $\mathcal{F}$ by means of a reduction of singularities $\mathcal{R}(\mathcal{F})$. The hypothesis and remark (9) applied to $\mathcal{F}$ imply $Det(\mathcal{F}) - \sum_p 2\delta_p(\mathcal{F}) < 0$. Let $\sharp\mathcal{R}(\mathcal{F})$ (resp. $\sharp\mathcal{R}(\mathcal{F}, p)$) denote the number of blow ups in $\mathcal{R}(\mathcal{F})$ (resp. in the reduction of $p \in Sing\mathcal{F}$).

Since the Euler number $c_2(M)$ increases by one under a blow up, we obtain from Baum-Bott's formula (4) § 2.1: $Det(\mathcal{F}) - \sum_p 2\delta_p(\mathcal{F}) = Det(\widetilde{\mathcal{F}}) - \sharp\mathcal{R}(\mathcal{F})$ and hence $\sharp Sing\widetilde{\mathcal{F}} - \sharp\mathcal{R}(\mathcal{F}) < 0$, where $\sharp Sing\widetilde{\mathcal{F}}$ denotes the number of singularities. Let D_p be the tree of rational curves introduced in the reduction of a singularity p. If we suppose by absurd that each singularity p of $\mathcal{F}$ has a finite number of local separatrices, then each component of D_p is $\widetilde{\mathcal{F}}$-invariant. Since D_p has $\sharp\mathcal{R}(\mathcal{F}, p) - 1$ intersection points of its components, then $\sharp(Sing\widetilde{\mathcal{F}} \cap D_p) \geq \sharp\mathcal{R}(\mathcal{F}, p) - 1$; but, in fact, the existence of a local separatrix at p (cf. [C-S]) implies $\sharp(Sing\widetilde{\mathcal{F}} \cap D_p) \geq \sharp\mathcal{R}(\mathcal{F}, p)$, that is, $\sharp Sing\widetilde{\mathcal{F}} - \sharp\mathcal{R}(\mathcal{F}) \geq 0$, a contradiction. □

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